Inner Product space

Def 1. Inner product space (Euclidean vector space). let E be a vector space over a field F. let to each pair of vectors $x, y \in E$ there be assigned a scalar $\langle x, y \rangle \in F$. The mapping is called an inner product in E if it satisfies the following axioms $\forall\ x_1, x_2, x, y \in E$ and $\alpha \in F$;

(i) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

(ii) $\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$

(iii) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

(iv) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ iff $x = 0$

The vector space E is called inner product space or Euclidean vector space.

2. **Length (Norm) of Element**: let E be an inner product space and $x \in E$. Then length or norm of x is denoted by $\|x\|$ and is defined as $\sqrt{\langle x, x \rangle}$ ie $\|x\| = \sqrt{\langle x, x \rangle}$.

$\|\alpha x\|^2 = \langle \alpha x, \alpha x \rangle = \alpha \bar{\alpha} \langle x, x \rangle = |\alpha|^2 \|x\|^2$

Hence $\|\alpha x\| = |\alpha| \|x\|$ $\forall\ \alpha \in F$ and $x \in E$.

1. Show that the following is an inner product in R^2:

$$\langle x, y \rangle = x_1 y_1 - x_1 y_2 - x_2 y_1 + 3 x_2 y_2$$

where $x = (x_1, x_2)$, $y = (y_1, y_2) \in R^2$

Sol: let $x = (x_1, x_2)$, $y = (y_1, y_2)$, $z = (z_1, z_2) \in R^2$ and $\alpha \in R$.

(I) Then $\langle x, y \rangle = x_1 y_1 - x_1 y_2 - x_2 y_1 + 3 x_2 y_2$

$= y_1 x_1 - y_1 x_2 - y_2 x_1 + 3 y_2 x_2$

$= \langle y, x \rangle$

II. $\langle x+y, z\rangle = \langle (x_1+y_1,\ x_2+y_2), (z_1, z_2)\rangle$

$= (x_1+y_1)z_1 - (x_1+y_1)z_2 - (x_2+y_2)z_1 + 3(x_2+y_2)z_2$

$= (x_1z_1 + y_1z_1) - (x_1z_2 + y_1z_2) - (x_2z_1 + y_2z_1) + 3(x_2z_2) + 3y_2z_2$

$= (x_1z_1 - x_1z_2 - x_2z_1 + 3x_2z_2) + (y_1z_1 - y_1z_2 - y_2z_1 + 3y_2z_2)$

$= \langle x, z\rangle + \langle y, z\rangle$

III $\langle \alpha x, y\rangle = \langle (\alpha x_1, \alpha x_2), (y_1, y_2)\rangle$

$= \alpha x_1y_1 - \alpha x_1y_2 - \alpha x_2y_1 + 3\alpha x_2y_2$

$= \alpha(x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2)$

$= \alpha\langle x, y\rangle$

IV $\langle x, x\rangle = x_1^2 - 2x_1x_2 + 3x_2^2$

$= (x_1^2 - 2x_1x_2 + x_2^2) + 2x_2^2 = (x_1 - x_2)^2 + 2x_2^2 \geq 0$

Also $\langle x, x\rangle = 0$ iff $x_1 = 0, x_2 = 0$ ie $x = 0$

Hence the given product is an inner product in R^2.

(2) let E be the inner product space of polynomials with inner product given by $\langle f, g\rangle = \int_0^1 f(t)\,g(t)\,dt$.

let $f(t) = t+2$

and $g(t) = t^2 - 2t - 3$

find (i) $\langle f, g\rangle$ (ii) $\|f\|$.

Sol. I $\langle f, g\rangle = \int_0^1 (t+2)(t^2-2t-3)\,dt = \left[\frac{t^4}{4} - \frac{7t^2}{2} - 6t\right]_0^1 = \frac{-37}{4}$

II $\|f\| = \sqrt{\langle f, f\rangle}$

Now $\langle f, f\rangle = \int_0^1 (t+2)(t+2)\,dt = \int_0^1 (t+2)^2\,dt = \left|\frac{(t+2)^3}{3}\right|_0^1 = \frac{19}{3}$

$\therefore \|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\frac{19}{3}}$ Ans.

Apply Gram Schmidt process to vectors $v_1 = (3,0,4)$, $v_2 = (-1,0,7)$ $v_3 = (2,9,11)$ of inner product space R^3 with usual inner product to obtain an orthonormal basis of R^3.

Sol:- let $u_1 = (3,0,4)$

Take $u_2 = v_2 - \dfrac{\langle v_2, u_1\rangle}{\|u_1\|^2} u_1$

$= (-1,0,7) - \dfrac{\langle(-1,0,7),(3,0,4)\rangle}{\|(3,0,4)\|^2}(3,0,4)$

$= (-1,0,7) - \dfrac{(-3+0+28)}{9+16}(3,0,4)$

$= (-1,0,7) - (3,0,4) = (-4,0,3)$.

Again $u_3 = v_3 - \dfrac{\langle v_3, u_1\rangle}{\|u_1\|^2}u_1 - \dfrac{\langle v_3, u_2\rangle}{\|u_2\|^2}u_2$.

$= (2,9,11) - \dfrac{\langle(2,9,11),(3,0,4)\rangle}{\|(3,0,4)\|^2}(3,0,4) - \dfrac{\langle(2,9,11),(-4,0,3)\rangle}{\|(-4,0,3)\|^2}(-4,0,3)$

$= (2,9,11) - \dfrac{(6+0+44)}{9+16}(3,0,4) - \dfrac{(-8+0+33)}{16+9}(-4,0,3)$

$= (2,9,11) - 2(3,0,4) - (-4,0,3) = (0,9,0)$.

The Required orthonormal basis of R^3 is

$e_1 = \dfrac{u_1}{\|u_1\|}$, $e_2 = \dfrac{u_2}{\|u_2\|}$, $e_3 = \dfrac{u_3}{\|u_3\|}$

ie $\{e_1, e_2, e_3\} = \left\{\left(\frac{3}{5}, 0, \frac{4}{5}\right), \left(-\frac{4}{5}, 0, \frac{3}{5}\right), (0,1,0)\right\}$

Ans

(4) Let E be the inner product space over R of polynomials in $R[x]$ of degree ≤ 2 with inner product

$$\langle f, g \rangle = \int_0^1 f(t)\, g(t)\, dt \quad , \quad f, g \in E$$

Apply Gram-Schmidt process to find orthonormal basis of V.

Sol: Let $\{1, t, t^2\}$ be the standard basis of E.

Take $u_1 = 1$

set $u_2 = t - \frac{\langle t, u_1 \rangle}{\|u_1\|^2} u_1 = t - \frac{\langle t, 1 \rangle}{1} \cdot 1$

$$= t - \int_0^1 t\,dt = t - \frac{1}{2}.$$

Again $u_3 = t^2 - \frac{\langle t^2, 1 \rangle}{\|u_1\|^2} u_1 - \frac{\langle t^2, t - \frac{1}{2} \rangle}{\|u_2\|^2} (t - \frac{1}{2})$

$$= t^2 - \int_0^1 t^2 dt - \frac{\int_0^1 t^2 (t - \frac{1}{2}) dt}{\int_0^1 (t - \frac{1}{2}) dt} (t - \frac{1}{2})$$

$$= t^2 - \frac{1}{3} - \frac{(\frac{1}{4} - \frac{1}{6})}{(\frac{1}{3} + \frac{1}{4} - \frac{1}{2})} (t - \frac{1}{2}) = t^2 - t + \frac{1}{6}$$

Now

$\|u_1\| = 1$

$\|u_2\| = \sqrt{\int_0^1 (t - \frac{1}{2})^2 dt} = \sqrt{\frac{1}{12}} = \frac{1}{2\sqrt{3}}$,

$\|u_3\| = \sqrt{\int_0^1 (t^2 - t + \frac{1}{6})^2 dt} = \frac{1}{\sqrt{180}} = \frac{1}{6\sqrt{5}}$

Hence $\left\{ 1, \sqrt{3}(2x-1), \sqrt{5}(6x^2 - 6x + 1) \right\}$ is the required orthonormal basis of E.

Find an orthonormal basis of the subspace of R^4 spanned by $v_1 = (1,1,1,1)$, $v_2 = (1,2,4,5)$, $v_3 = (1,-3,-4,-2)$.

Sol: Here

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 5 \\ 1 & -3 & -4 & -2 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & -4 & -5 & -3 \end{bmatrix}$$

$R_3 \to R_3 + 4R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 7 & 13 \end{bmatrix}.$$

$\therefore$ Rank of the matrix $= 3$

Thus $\{v_1, v_2, v_3\}$ is L.I. set

we shall apply Gram-Schmidt orthogonalization process to find orthonormal basis.

set $u_1 = v_1 = (1,1,1,1)$

$$u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\|u_1\|^2} u_1 = (1,2,4,5) - \frac{\langle (1,2,4,5), (1,1,1,1) \rangle}{\|(1,1,1,1)\|^2}(1,1,1,1)$$

$$= (1,2,4,5) - \frac{1+2+4+5}{1+1+1+1}(1,1,1,1)$$

$$= (1,2,4,5) - 3(1,1,1,1)$$

$$= (-2,-1,1,2)$$

$$u_3 = v_3 - \frac{\langle v_3, u_1 \rangle}{\|u_1\|^2} u_1 - \frac{\langle v_3, u_2 \rangle}{\|u_2\|^2} u_2$$

$$= (1,-3,-4,2) - \frac{\langle (1,-3,-4,-2), (1,1,1,1) \rangle}{\|(1,1,1,1)\|^2}(1,1,1,1) - \frac{\langle (1,-3,-4,-2)\ (-2,-1,1,2) \rangle}{\|(-2,-1,1,2)\|^2}(-2,-1,1,2)$$

$$= (1,-3,-4,-2) - \frac{(-8)}{4}(1,1,1,1) - \frac{(-7)}{10}(-2,-1,1,2)$$

$$= \left(\frac{8}{5}, \frac{-17}{10}, \frac{-13}{10}, \frac{7}{5}\right)$$

$\therefore$ Required orthonormal basis is $\{e_1, e_2, e_3\}$ where

$$e_1 = \frac{u_1}{\|u_1\|}, \qquad e_2 = \frac{u_2}{\|u_2\|}, \qquad e_3 = \frac{u_3}{\|u_3\|}$$

ie $e_1 = \frac{1}{2}(1,1,1,1)$, $\quad e_2 = \frac{1}{\sqrt{10}}(-2,-1,1,2)$, $\quad e_3 = \frac{1}{\sqrt{910}}(16,-17,-13,14)$